

GERBAL CENTRAL EXTENSIONS OF REDUCTIVE GROUPS BY \mathcal{K}_3

PAVEL SAFRONOV

ABSTRACT. We classify central extensions of a reductive group G by \mathcal{K}_3 and BK_3 , the sheaf of third Quillen K-theory groups and its classifying stack. These turn out to be parametrized by the group of Weyl-invariant quadratic forms on the cocharacter lattice valued in k^\times and the group of integral Weyl-invariant cubic forms on the cocharacter lattice respectively.

INTRODUCTION

Let G be a reductive group over a field and let $\mathrm{LG} = G((t))$ be its loop group. In representation theory one is often interested in central extensions of LG . A central extension of LG by \mathbf{G}_m is the same as a multiplicative torsor [SGA7, Exp. 7, Section 1] on LG , i.e. a torsor \mathcal{T} together with an identification $\mathcal{T}_{xy} \cong \mathcal{T}_x \otimes \mathcal{T}_y$ for any two points $x, y \in \mathrm{LG}$. Given a geometric object on the loop space LG one can wonder whether it comes from the group G itself as G is a much easier geometric object than LG : it is an affine scheme of finite type in contrast to LG being an ind-scheme.

Indeed, there is a transgression map [KV]

$$H^1(G, \mathcal{K}_2) \rightarrow H^1(\mathrm{LG}, \mathcal{O}^\times),$$

where \mathcal{K}_2 is the sheaf of second Quillen K-theory groups, which sends a \mathcal{K}_2 -torsor on G to a \mathbf{G}_m -torsor on LG . In fact, it behaves well with respect to multiplicative structures and so it sends a \mathcal{K}_2 -central extension of G to a \mathbf{G}_m -central extension of LG .

Thus, one is naturally led to consider \mathcal{K}_2 -central extensions of G . Such a classification has been performed by Esnault–Kahn–Levine–Viehweg [EKLIV] and Brylinski–Deligne [BrD].

Brylinski and Deligne construct a spectral sequence which computes the \mathcal{K}_n -cohomology of G from the \mathcal{K}_n -cohomology of its maximal torus, a much simpler task. Another spectral sequence allows one to classify \mathcal{K}_n -central extensions of G from the knowledge of \mathcal{K}_n -cohomology of G .

For $n = 2$ both spectral sequences degenerate after the first nontrivial page. This allowed Brylinski and Deligne to obtain the following classification:

Theorem (Brylinski–Deligne). *Let G be a split connected reductive group over a field with a simply-connected derived group. Then the group of \mathcal{K}_2 -central extensions of G is isomorphic to the group of Weyl-invariant quadratic forms on the cocharacter lattice.*

Studying double loop groups $\mathrm{L}^2G = G((t))((s))$, one naturally encounters not ordinary central extensions, but gerbal extensions, i.e. central extensions by the commutative group stack BG_m [AK], [FZ]. Since we have to do the transgression map twice, \mathbf{G}_m -central extensions of L^2G come from \mathcal{K}_3 -central extensions of G . The goal of this paper is to classify such extensions. We obtain the following result (Theorem 4.2).

Key words and phrases. K-cohomology, central extensions, gerbes.

Theorem. *Let G be a split connected reductive group over a field k with a simply-connected derived group. Then the group of \mathcal{K}_3 -central extensions of G is isomorphic to the group of Weyl-invariant quadratic forms on the cocharacter lattice valued in k^\times . The group of gerbal central extensions of G by \mathcal{K}_3 is isomorphic to the group of integral Weyl-invariant cubic forms on the cocharacter lattice.*

We show that using our methods one can compute $\mathrm{CH}^3(G)$, the third Chow group of codimension 3 cycles on the group G . It turns out to be isomorphic to $(\mathbf{Z}/2\mathbf{Z})^{n_{\mathrm{BDG}}}$, where n_{BDG} is the number of times a G_2 , B_3 or D_4 subdiagram appears in the Dynkin diagram of G . Thus, $\mathrm{CH}^3(G)$ is 2-torsion unless G only contains type A and C subgroups in which case it is trivial. Note that one has $\mathrm{CH}^1(G) \equiv \mathrm{Pic}(G) = 0$ and $\mathrm{CH}^2(G) = 0$ for all G satisfying our assumptions. The full Chow ring of algebraic groups was known before due to results of Borel, Chevalley, Grothendieck [Gr], Kac [Kac], Marlin [Mar], Kaji–Nakagawa [KN] and many others.

We have concentrated on the simply-connected case for simplicity; the general case has both combinatorial difficulties and the E_1 degeneration of the Brylinski–Deligne spectral sequence is not immediate. The results generalize to split groups over more general regular bases over a field, but a generalization to the non-split case is unclear. Our statements could be useful for a partial computation of \mathcal{K} -cohomology $H^\bullet(G, \mathcal{K}_n)$ for $n > 3$.

The paper is organized in the following way. We begin with a recollection on \mathcal{K} -cohomology of tori computed in [EKL] and [BrD]. We recall a classification of \mathcal{K}_2 -classes on tori via alternating forms together with quadratic refinements and prove a version of this classification for \mathcal{K}_3 -classes on tori. Section 2 contains the core technical results. We recall the spectral sequence introduced by Brylinski and Deligne for the computation of $H^\bullet(G, \mathcal{K}_n)$ and compute it for $n = 2, 3$ for a semisimple group G . The spectral sequence degenerates at E_1 in both cases. In Section 3 we combine the results of Sections 1 and 2 to obtain \mathcal{K}_3 -cohomology of reductive groups G . In the final section we use these results to obtain a classification of multiplicative \mathcal{K}_3 -torsors and gerbes on the group and thus the corresponding central extensions. We end with an explanation of the relation between \mathcal{K}_n -extensions of G and \mathbf{G}_m -extensions of its iterated loop groups.

Acknowledgements. The author would like to thank Xinwen Zhu for a useful conversation, Burt Totaro for some comments on the draft and the referee for improving the exposition.

1. \mathcal{K} -COHOMOLOGY OF TORI

Let H be a split torus over a field k . We denote by $X = \mathrm{Hom}(H, \mathbf{G}_m)$ its group of characters and by $Y = \mathrm{Hom}(\mathbf{G}_m, H)$ the dual group of cocharacters. Let \mathcal{K}_n be the Zariski sheafification of the presheaf of Quillen K-theory groups. For instance, $\mathcal{K}_1 \cong \mathcal{O}^\times$ and we denote by $\{f\}$ the element of \mathcal{K}_1 corresponding to the invertible function $f \in \mathcal{O}^\times$. The sheaves \mathcal{K}_\bullet form a graded ring and we denote the multiplication in K-theory by a period.

We recall a computation of the \mathcal{K} -cohomology of H due to Brylinski and Deligne [BrD, Lemma 3.3.1]:

Proposition 1.1 (Brylinski–Deligne). *The graded ring $H^0(H, \mathcal{K}_\bullet)$ is generated by the graded ring $\mathbf{K}_\bullet(k)$ and X in degree 1 subject to the following relations:*

- (1) $X \rightarrow H^0(H, \mathcal{K}_1)$ is additive,

(2) $x.x = x.\{-1\}$ for every $x \in X$.

Moreover, the higher cohomology vanishes.

From the presentation we see that there is an increasing filtration V on $H^0(H, \mathcal{K}_\bullet)$ compatible with the grading where $V_n H^0(H, \mathcal{K}_\bullet)$ is the subgroup of elements of $H^0(H, \mathcal{K}_\bullet)$ containing not more than n factors of X . The associated graded ring is given by

$$\mathrm{gr}_n^V H^0(H, \mathcal{K}_m) = \bigwedge^n X \otimes K_{m-n}(k).$$

Note that $K_m(k) \subset H^0(H, \mathcal{K}_m)$ is a direct summand, so we also have an embedding $X \otimes K_{m-1}(k) \hookrightarrow H^0(H, \mathcal{K}_m)$ given by the multiplication in K -theory.

The filtration on $H^0(H, \mathcal{K}_2)$ has a direct summand $K_2(k)$ and thus the reduced cohomology group $\tilde{H}^0(H, \mathcal{K}_2)$ fits into an exact sequence

$$0 \rightarrow X \otimes k^\times \rightarrow \tilde{H}^0(H, \mathcal{K}_2) \rightarrow \wedge^2 X \rightarrow 0.$$

This extension has the following description.

Proposition 1.2 (Brylinski–Deligne). *The reduced cohomology group $\tilde{H}^0(H, \mathcal{K}_2)$ is isomorphic to the group of alternating forms A on the cocharacter lattice together with a quadratic refinement $q: Y \rightarrow k^\times$ obeying*

$$\frac{q(x+y)}{q(x)q(y)} = (-1)^{A(x,y)}.$$

Note that $q: Y \rightarrow k^\times$ is not a map of groups unless A is zero. We will also need an analog of this proposition for \mathcal{K}_3 -classes.

Proposition 1.3. *The reduced cohomology $\tilde{H}^0(H, \mathcal{K}_3)$ is isomorphic to the group of totally antisymmetric 3-forms $A(-, -, -)$ on the cocharacter lattice together with two refinements $q_1(-, -): Y \times Y \rightarrow k^\times$ and $q_2(-): Y \rightarrow K_2(k)$ satisfying*

$$\begin{aligned} q_1(x, y) &= q_1(y, x)^{-1}, \\ q_1(x + y, z) &= q_1(x, z)q_1(y, z)(-1)^{A(x, y, z)}, \\ q_2(x + y) &= q_2(x).q_2(y).\{-1, q_1(x, y)\}. \end{aligned}$$

Proof. If $\{e_i\}$ denotes a basis of X , a general element of $\tilde{H}^0(H, \mathcal{K}_3)$ can be represented as

$$a_{ijk}e_i.e_j.e_k + e_i.e_j.\{f_{ij}\} + e_i.g_i$$

for some integer matrix a_{ijk} , a collection f_{ij} of elements of k^\times and a collection g_i of elements of $K_2(k)$.

We define

$$a \in X^{\otimes 3}, \quad f \in X \otimes X \otimes k^\times, \quad g \in X \otimes K_2(k)$$

by a linear extension of these collections.

Any two representations of the same element are connected by a sequence of the following three identifications:

$$(1) \quad \begin{aligned} a_{ijk} &\sim a_{ijk} + c_i c_j, & f_{ij} &\sim f_{ij}(-1)^{c_i}, & g_i &\sim g_i, \\ a_{ijk} &\sim a_{ijk} + c_j c_k, & f_{ij} &\sim f_{ij}(-1)^{c_j}, & g_i &\sim g_i, \\ a_{ijk} &\sim a_{ijk}, & f_{ij} &\sim f_{ij} h^{c_i c_j}, & g_i &\sim g_i \cdot \{(-1)^{c_i}, h\}. \end{aligned}$$

Here c_i is a collection of integer numbers and $h \in k^\times$.

We define

$$A(x, y, z) = a(x, y, z) - a(y, x, z) - a(x, z, y) + a(y, z, x) + a(z, x, y) - a(z, y, x),$$

the antisymmetrization of a . It is clearly well-defined for an element of $\tilde{H}^0(H, \mathcal{K}_3)$.

Next, we define

$$q_1(x, y) = \frac{f(x, y)}{f(y, x)} (-1)^{a(x, x, y) + a(x, y, x) + a(y, x, x) + a(x, y, y) + a(y, x, y) + a(y, y, x)}.$$

Again, a straightforward check shows that it is invariant under transformations (1).

The form q_1 is obviously alternating. It is not linear in the arguments, but satisfies the following equations:

$$\begin{aligned} q_1(x_1 + x_2, y) &= q_1(x_1, y) q_1(x_2, y) (-1)^{A(x_1, x_2, y)}, \\ q_1(x, y_1 + y_2) &= q_1(x, y_1) q_1(x, y_2) (-1)^{A(x, y_1, y_2)}. \end{aligned}$$

Finally, we define

$$q_2(x) = g(x) \cdot \{-1, f(x, x)\}.$$

It satisfies the following equation:

$$\begin{aligned} q_2(x + y) &= g(x + y) \cdot \{-1, f(x + y, x + y)\} \\ &= g(x) \cdot g(y) \cdot \{-1, f(x, x) f(x, y) f(y, x) f(y, y)\} \\ &= q_2(x) q_2(y) \cdot \{-1, f(x, y) f(y, x)\} \\ &= q_2(x) \cdot q_2(y) \cdot \{-1, q_1(x, y)\}. \end{aligned}$$

Let us now prove that the data of A , q_1 and q_2 satisfying the linearity equations above is equivalent to specifying an element of $\tilde{H}^0(H, \mathcal{K}_3)$.

Indeed, we have defined a surjective map $\tilde{H}^0(H, \mathcal{K}_3) \twoheadrightarrow \wedge^3 X$. Its kernel K is represented by elements with a_{ijk} whose antisymmetrization is zero. Such a matrix can be made zero using the first two transformations in (1). So, elements of K can be represented as

$$e_i \cdot e_j \cdot \{f_{ij}\} + e_i \cdot g_i.$$

We have defined a surjective map $K \twoheadrightarrow \wedge^2 X \otimes k^\times$. Its kernel consists of elements with f_{ij} symmetric. Using the last transformation in (1) we make $f_{ij} = 1$, so we get an exact sequence

$$0 \rightarrow X \otimes K_2(k) \rightarrow K \rightarrow \wedge^2 X \otimes k^\times \rightarrow 0.$$

Note that the filtration

$$X \otimes K_2(k) \subset K \subset \tilde{H}^0(H, \mathcal{K}_3)$$

we have defined during the proof coincides with the V filtration. □

2. \mathcal{K} -COHOMOLOGY OF SEMISIMPLE GROUPS

In this section we let G be a split connected simply-connected semisimple group over a field k , $H \subset G$ is a split maximal torus and W the Weyl group. Let $W^{(n)} \subset W$ be the subset of elements of W of length $l(w_0) - n$, where $l(w_0)$ is the length of the longest element. Let X be the character group of H and Y the cocharacter group. Let $s_i \in W$ denote the simple reflections with respect to a chosen Borel in G and $\alpha_i^\vee \in Y$ are the simple coroots. Since G is simply-connected, the weight lattice coincides with X .

2.1. Brylinski–Deligne spectral sequence. The Bruhat decomposition of G allows one to compute the cohomology $H^\bullet(G, \mathcal{K}_n)$ via the Cousin complex $C^\bullet(G, \mathcal{K}_n)$, where each term is expressed in terms of $H^0(H, \mathcal{K}_j)$ for some j . The filtration on $H^0(H, \mathcal{K}_j)$ extends to a filtration on $C^\bullet(G, \mathcal{K}_n)$. Thus, Brylinski and Deligne arrive at a spectral sequence of the filtered complex whose E_0 page has terms

$$E_0^{p,q} = \bigoplus_{w \in W^{(p+q)}} \bigwedge^{-2p-q} X \otimes \mathcal{K}_{n+p}(k).$$

Note that since the filtration on $H^0(H, \mathcal{K}_j)$ is increasing, the cohomological spectral sequence is concentrated in the second quadrant.

The E_0 differential $d: E_0^{p,q} \rightarrow E_0^{p,q+1}$ is given by the Bruhat order and has the following description. It is enough to specify the differential d by its components

$$d_w^{w_1}: \bigwedge^{-2p-q} X \otimes \mathcal{K}_{n+p}(k) \rightarrow \bigwedge^{-2p-q-1} X \otimes \mathcal{K}_{n+p}(k),$$

where the first term sits at $w \in W^{(p+q)}$ and the second term sits at $w_1 \in W^{(p+q+1)}$. If $w = w'w''$ and $w_1 = w's_iw''$ for a simple reflection s_i , then the differential $d_w^{w_1}$ is given by contracting the $\bigwedge^{-2p-q} X$ part with $(w'')^{-1}(\alpha_i^\vee)$.

The spectral sequence converges to the \mathcal{K} -cohomology of G :

$$E_0^{p,q} \Rightarrow H^{p+q}(G, \mathcal{K}_n).$$

2.2. \mathcal{K}_2 -cohomology. Let us illustrate the power of the spectral sequence by computing the \mathcal{K}_2 -cohomology of G . We will outline the proof of the following theorem [BrD, Proposition 4.6]:

Theorem 2.1 (Brylinski–Deligne). *Let G be a split connected simply-connected semisimple group over a field k . It has the following \mathcal{K}_2 -cohomology groups:*

- $H^0(G, \mathcal{K}_2) = \mathcal{K}_2(k)$,
- $H^1(G, \mathcal{K}_2)$ is isomorphic to the group of W -invariant quadratic forms on the cocharacter lattice.

The higher cohomology groups vanish.

The set $W^{(0)}$ has only one element, the longest element w_0 . The set $W^{(1)}$ can be identified with the set of simple roots as $w_0 s_i \in W^{(1)}$ for any s_i a simple reflection. Finally, $W^{(2)}$ consists of elements $w_0 s_i s_j$ for any two unequal simple roots s_i and s_j ; the elements $w_0 s_i s_j$ and $w_0 s_j s_i$ are identified if α_i and α_j are orthogonal.

The E_0 page is

$$\begin{array}{ccccc}
\oplus_{W(2)} \mathbf{Z} & 0 & 0 \\
\uparrow & & \\
\oplus_{W(1)} X & 0 & 0 \\
\uparrow & & \\
\bigwedge^2 X & \oplus_{W(1)} k^\times & 0 \\
& \uparrow & \\
0 & X \otimes k^\times & 0 \\
& & \\
0 & 0 & K_2(k)
\end{array}$$

Here p labels the horizontal axis and q the vertical axis. The whole spectral sequence is concentrated in the second quadrant with $E_0^{0,0} = K_2(k)$.

We can identify $\oplus_{W(1)} \mathbf{Z}$ with the weight lattice: since G is simply-connected, the map $X \rightarrow \oplus_{W(1)} \mathbf{Z}$ sending $x \mapsto \{x(\alpha_i^\vee)\}$ is an isomorphism. Therefore, $E_1^{-1,1} = E_1^{-1,2} = 0$. The same argument shows that the differential $E_0^{-p,p} \rightarrow E_0^{-p,p+1}$ is injective, hence $E_1^{-p,p} = 0$.

Proposition 2.1. *The differential $E_0^{-2,3} \rightarrow E_0^{-2,4}$ is surjective.*

Proof. See the second half of the proof of [BrD, Proposition 4.6]. \square

Therefore, $E_1^{-2,4} = 0$. Finally, we can identify $E_1^{-2,3}$ with the group of bilinear forms C on Y satisfying

$$C(\alpha_i^\vee, \alpha_j^\vee) + C(\alpha_j^\vee, s_j(\alpha_i^\vee)) = 0,$$

or, equivalently, with the group of W -invariant quadratic forms $\{y \mapsto C(y, y)\}$ on the cocharacter lattice.

The E_1 page of the spectral sequence is

$$\begin{array}{ccc}
0 & 0 & 0 \\
E_1^{-2,3} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & K_2(k).
\end{array}$$

The spectral sequence degenerates at E_1 , thus we obtain the desired cohomology groups.

2.3. \mathcal{K}_3 -cohomology. Let us apply the spectral sequence for the computation of the cohomology groups $H^n(G, \mathcal{K}_3)$. We will prove the following theorem:

Theorem 2.2. *Let G be a split connected simply-connected semisimple group over a field k . Then it has the following \mathcal{K}_3 -cohomology:*

- $H^0(G, \mathcal{K}_3) \cong K_3(k)$,
- $H^1(G, \mathcal{K}_3)$ is isomorphic to the group of W -invariant quadratic forms on the cocharacter lattice with values in k^\times ,
- $H^2(G, \mathcal{K}_3)$ is isomorphic to the group of W -invariant cubic forms on the cocharacter lattice,

- $\mathrm{CH}^3(G) \cong \mathrm{H}^3(G, \mathcal{K}_3)$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^{n_{\mathrm{BDG}}}$, where n_{BDG} is the number of times a G_2 , B_3 or D_4 subdiagram appears in the Dynkin diagram of G .

The higher cohomology groups vanish.

The groups $\mathrm{H}^0(G, \mathcal{K}_n)$ and $\mathrm{H}^1(G, \mathcal{K}_n)$ for all n have been previously computed by Gille [Gil, Theorem B].

We will need a description of $W^{(3)}$. Its elements are $w_0 s_i s_j s_k$, where the length of $s_i s_j s_k$ is 3. The following statement can be obtained from the presentation of the Weyl group as a Coxeter group.

Lemma 2.1. *The product of simple reflections $s_i s_j s_k$ has length 1 iff any of the following is satisfied:*

- The adjacent indices are equal.
- The adjacent roots are orthogonal and $i = k$.

Two length 3 elements $s_i s_j s_k$ and $s_l s_m s_n$ are equal iff any of the following holds:

- (ijk) is obtained from (lmn) by a permutation of orthogonal roots
- The roots i and j are connected by a single edge and $i = k = m$, $j = l = n$.

The spectral sequence computing $\mathrm{H}^\bullet(G, \mathcal{K}_3)$ has the following E_0 page:

$$\begin{array}{ccccccc}
 \oplus_{W^{(3)}} \mathbf{Z} & & 0 & & 0 & & 0 \\
 \uparrow & & & & & & \\
 \oplus_{W^{(2)}} X & & 0 & & 0 & & 0 \\
 \uparrow & & & & & & \\
 \oplus_{W^{(1)}} \bigwedge^2 X & \oplus_{W^{(2)}} k^\times & & 0 & & & 0 \\
 \uparrow & \uparrow & & & & & \\
 \bigwedge^3 X & \oplus_{W^{(1)}} X \otimes k^\times & & 0 & & & 0 \\
 & \uparrow & & & & & \\
 0 & \bigwedge^2 X \otimes k^\times & \oplus_{W^{(1)}} K_2(k) & & 0 & & \\
 & & \uparrow & & & & \\
 0 & 0 & X \otimes K_2(k) & & 0 & & \\
 & & & & & & \\
 0 & 0 & 0 & & 0 & & K_3(k)
 \end{array}$$

Due to simply-connectivity of G we again have

$$E_1^{-1,1} = E_1^{-1,2} = 0$$

and the differentials $E_0^{-2,2} \rightarrow E_0^{-2,3}$ and $E_0^{-3,3} \rightarrow E_0^{-3,4}$ are injective.

As explained in the previous section, $E_1^{-2,4} = 0$ and $E_1^{-2,3}$ is isomorphic to the group of W -invariant quadratic forms on the cocharacter lattice with values in k^\times .

Proposition 2.2. *The cohomology group $E_1^{-3,4}$ is zero.*

Proof. We will represent elements of $\oplus_{W(1)} \wedge^2 X$ by antisymmetric forms $D_i(-, -)$ on Y . The closed elements in $E_0^{-3,4}$ are antisymmetric forms D_i that satisfy

$$D_i(\alpha_j^\vee, \alpha_k^\vee) + D_j(s_j(\alpha_i^\vee), \alpha_k^\vee) = 0$$

for every coroots $\alpha_i^\vee, \alpha_j^\vee, \alpha_k^\vee$. The form $D_i(-, -)$ is exact if its trilinear extension $D(-, -, -)$ is totally antisymmetric.

We expand the closedness equation as

$$D_i(\alpha_j^\vee, \alpha_k^\vee) + D_j(\alpha_i^\vee, \alpha_k^\vee) - \alpha_j(\alpha_i^\vee)D_j(\alpha_j^\vee, \alpha_k^\vee) = 0.$$

Therefore, $D_i(\alpha_j^\vee, \alpha_k^\vee) = -D_j(\alpha_i^\vee, \alpha_k^\vee)$ if α_i and α_j are orthogonal.

To show that $D(-, -, -)$ is antisymmetric in the first two entries for any arguments we just need to show that $D_j(\alpha_j^\vee, \alpha_k^\vee) = 0$ for any α_j^\vee and α_k^\vee .

Indeed,

$$\begin{aligned} D_j(\alpha_j^\vee, \alpha_k^\vee) &= -D_j(\alpha_k^\vee, \alpha_j^\vee) \\ &= D_k(\alpha_j^\vee, \alpha_j^\vee) - \alpha_j(\alpha_k^\vee)D_j(\alpha_j^\vee, \alpha_j^\vee) = 0. \end{aligned}$$

Here in the first line we have used the antisymmetry of D_j and in the second line we have used the closedness condition.

Therefore, any closed form D is antisymmetric in the first two and the last two arguments, hence it is totally antisymmetric. \square

Proposition 2.3. *The cohomology $E_1^{-3,6}$ is $(\mathbf{Z}/2\mathbf{Z})^{n_{\text{BDG}}}$, where n_{BDG} is the number of times a subdiagram of type G_2 , B_3 or D_4 appears in the Dynkin diagram of G .*

Proof. Suppose that one is given a collection of integers $\{d_{ijk}\} \in E_0^{-3,6}$, where i, j, k parametrize the coroots, defined for unequal adjacent indices satisfying

- $d_{ijk} = d_{jik}$ if i is orthogonal to j ,
- $d_{ijk} = d_{ikj}$ if j is orthogonal to k ,
- $d_{iji} = d_{jij}$ if i and j are connected by a single edge.

Note, that d_{iji} is undefined if α_i^\vee and α_j^\vee are orthogonal.

The collection $\{d_{ijk}\} \in E_0^{-3,6}$ is exact if the system

(2) $d_{ijk} = \phi_{ij}(\alpha_k^\vee) + \phi_{ik}(\alpha_j^\vee - \alpha_k(\alpha_j^\vee)\alpha_k^\vee) + \phi_{jk}(\alpha_i^\vee - \alpha_j(\alpha_i^\vee)\alpha_j^\vee - \alpha_k(\alpha_i^\vee)\alpha_k^\vee + \alpha_k(\alpha_j^\vee)\alpha_j(\alpha_i^\vee)\alpha_k^\vee)$ has a solution for some elements $\phi_{ij} \in X$ satisfying $\phi_{ij} = \phi_{ji}$ for orthogonal roots. Note that $\phi_{ii} = 0$.

First, consider a pair of simple roots i, j . From the classification of semisimple groups we have the following cases:

- $\alpha_j(\alpha_i^\vee) = 0$, i.e. the roots i and j are orthogonal. In this case we let

$$\phi_{ij}(\alpha_i^\vee) = \phi_{ji}(\alpha_i^\vee) = 0.$$

- $\alpha_j(\alpha_i^\vee) = \alpha_i(\alpha_j^\vee) = -1$, i.e. the roots i and j are connected by a single edge. Then we must satisfy the relation

$$(3) \quad d_{iji} = \phi_{ij}(\alpha_i^\vee) + \phi_{ji}(\alpha_j^\vee).$$

This obviously has a solution for any d_{iji} . Note that the relation coming from d_{jij} is the same one.

- $\alpha_j(\alpha_i^\vee) = -2$ and $\alpha_i(\alpha_j^\vee) = -1$, i.e. the roots i and j are connected by a double edge. We must satisfy the relations

$$(4) \quad \begin{aligned} d_{iji} &= \phi_{ij}(\alpha_i^\vee) + \phi_{ji}(\alpha_i^\vee) + 2\phi_{ji}(\alpha_j^\vee) \\ d_{jij} &= \phi_{ji}(\alpha_j^\vee) + \phi_{ij}(\alpha_j^\vee) + \phi_{ij}(\alpha_i^\vee). \end{aligned}$$

This system has a unique solution for any $\phi_{ji}(\alpha_i^\vee)$ and $\phi_{ij}(\alpha_j^\vee)$:

$$\begin{aligned} \phi_{ji}(\alpha_j^\vee) &= d_{iji} - d_{jij} + \phi_{ij}(\alpha_j^\vee) - \phi_{ji}(\alpha_i^\vee) \\ \phi_{ij}(\alpha_i^\vee) &= 2d_{jij} - d_{iji} - 2\phi_{ij}(\alpha_j^\vee) + \phi_{ji}(\alpha_i^\vee). \end{aligned}$$

- $\alpha_j(\alpha_i^\vee) = -3$ and $\alpha_i(\alpha_j^\vee) = -1$, i.e. the roots i and j are connected by a triple edge. We must satisfy the relations

$$(5) \quad \begin{aligned} d_{iji} &= \phi_{ij}(\alpha_i^\vee) + 2\phi_{ji}(\alpha_i^\vee) + 3\phi_{ji}(\alpha_j^\vee) \\ d_{jij} &= \phi_{ji}(\alpha_j^\vee) + 2\phi_{ij}(\alpha_j^\vee) + \phi_{ij}(\alpha_i^\vee). \end{aligned}$$

This system has a unique solution for any $\phi_{ji}(\alpha_i^\vee)$ and $\phi_{ij}(\alpha_j^\vee)$ if $d_{iji} - d_{jij}$ is even and it has no solutions if the difference $d_{iji} - d_{jij}$ is odd. Therefore, the elements with $d_{iji} - d_{jij}$ odd represent a nontrivial cohomology class if $\alpha_j(\alpha_i^\vee) = -3$.

Let us now see what happens for three distinct simple roots i, j, k . There are four cases depending on the number of pairs of orthogonal roots among i, j, k :

- All three roots are orthogonal. Then we have to solve the equation

$$(6) \quad d_{ijk} = \phi_{ij}(\alpha_k^\vee) + \phi_{ik}(\alpha_j^\vee) + \phi_{jk}(\alpha_i^\vee).$$

It clearly has a solution.

- There are two orthogonal pairs which we assume to be (j, k) and (i, k) . Then we have to solve the system

$$(7) \quad \begin{aligned} d_{ijk} &= \phi_{ij}(\alpha_k^\vee) + \phi_{ik}(\alpha_j^\vee) + \phi_{jk}(\alpha_i^\vee) - \alpha_j(\alpha_i^\vee)\phi_{jk}(\alpha_j^\vee) \\ d_{kji} &= \phi_{jk}(\alpha_i^\vee) + \phi_{ik}(\alpha_j^\vee) + \phi_{ji}(\alpha_k^\vee) - \alpha_i(\alpha_j^\vee)\phi_{ik}(\alpha_i^\vee) \end{aligned}$$

for the four variables $\phi_{ij}(\alpha_k^\vee), \phi_{ik}(\alpha_j^\vee), \phi_{jk}(\alpha_i^\vee), \phi_{ji}(\alpha_k^\vee)$. It again obviously has a solution.

- There is a single pair of orthogonal roots which we assume to be (i, k) . Then we have to solve the system

$$(8) \quad \begin{aligned} d_{ijk} &= \phi_{ij}(\alpha_k^\vee) + \phi_{ik}(\alpha_j^\vee) + \phi_{jk}(\alpha_i^\vee) - \alpha_k(\alpha_j^\vee)\phi_{ik}(\alpha_k^\vee) - \alpha_j(\alpha_i^\vee)\phi_{jk}(\alpha_j^\vee) + \alpha_k(\alpha_j^\vee)\alpha_j(\alpha_i^\vee)\phi_{jk}(\alpha_k^\vee) \\ d_{ikj} &= \phi_{ik}(\alpha_j^\vee) + \phi_{ij}(\alpha_k^\vee) + \phi_{kj}(\alpha_i^\vee) - \alpha_j(\alpha_k^\vee)\phi_{ij}(\alpha_j^\vee) - \alpha_j(\alpha_i^\vee)\phi_{kj}(\alpha_j^\vee) \\ d_{jki} &= \phi_{jk}(\alpha_i^\vee) + \phi_{ji}(\alpha_k^\vee) + \phi_{ik}(\alpha_j^\vee) - \alpha_k(\alpha_j^\vee)\phi_{ik}(\alpha_k^\vee) - \alpha_i(\alpha_j^\vee)\phi_{ik}(\alpha_i^\vee) \\ d_{kji} &= \phi_{kj}(\alpha_i^\vee) + \phi_{ik}(\alpha_j^\vee) + \phi_{ji}(\alpha_k^\vee) - \alpha_i(\alpha_j^\vee)\phi_{ik}(\alpha_i^\vee) - \alpha_j(\alpha_k^\vee)\phi_{ji}(\alpha_j^\vee) + \alpha_i(\alpha_j^\vee)\alpha_j(\alpha_k^\vee)\phi_{ji}(\alpha_i^\vee) \end{aligned}$$

for the five variables $\phi_{ij}(\alpha_k^\vee), \phi_{ik}(\alpha_j^\vee), \phi_{jk}(\alpha_i^\vee), \phi_{kj}(\alpha_i^\vee), \phi_{ji}(\alpha_k^\vee)$.

This system has a solution iff

$$(9) \quad \begin{aligned} d_{ijk} + d_{kji} + \alpha_j(\alpha_i^\vee)\phi_{jk}(\alpha_j^\vee) - \alpha_k(\alpha_j^\vee)\alpha_j(\alpha_i^\vee)\phi_{jk}(\alpha_k^\vee) + \alpha_j(\alpha_k^\vee)\phi_{ji}(\alpha_j^\vee) - \alpha_i(\alpha_j^\vee)\alpha_j(\alpha_k^\vee)\phi_{ji}(\alpha_i^\vee) \\ = d_{ikj} + d_{jki} + \alpha_j(\alpha_k^\vee)\phi_{ij}(\alpha_j^\vee) + \alpha_j(\alpha_i^\vee)\phi_{kj}(\alpha_j^\vee). \end{aligned}$$

- None of the roots are orthogonal. This cannot happen as it would imply that the Dynkin diagram of G has a 2-cycle.

To construct a solution $\phi_{ij}(-)$, we run the following algorithm. For each three orthogonal roots i, j, k solve (6) for $\phi_{ij}(\alpha_k^\vee)$.

Decompose the Dynkin diagram of G into connected components. For each component we do the following. First, for each pair of adjacent roots i, j in the diagram and any other orthogonal root k we solve (7) for $\phi_{ij}(\alpha_k^\vee)$. We set $\phi_{ij}(\alpha_j^\vee) = 0$. Then depending on the component type we define $\phi_{ij}(\alpha_k^\vee)$ in the following way.

Type A_n ($n \geq 3$). Linearly order the simple roots. For each triple of simple roots $i, j = i + 1, k = i + 2$ we have to solve

$$\begin{aligned} d_{ijk} + d_{kji} - \phi_{jk}(\alpha_j^\vee) - \phi_{ji}(\alpha_j^\vee) &= d_{ikj} + d_{jki} \\ d_{iji} &= \phi_{ij}(\alpha_i^\vee) + \phi_{ji}(\alpha_j^\vee) \\ d_{jkj} &= \phi_{jk}(\alpha_j^\vee) + \phi_{kj}(\alpha_k^\vee). \end{aligned}$$

Set $\phi_{21}(\alpha_2^\vee) = 0$. Then for each triple $i, j = i + 1$ and $k = i + 2$ of simple roots in this component we can iteratively solve the system as follows. From the first equation one uniquely determines $\phi_{jk}(\alpha_j^\vee)$. From the last two equations one uniquely determines $\phi_{ij}(\alpha_i^\vee)$ and $\phi_{kj}(\alpha_k^\vee)$.

Type B_n ($n \geq 3$). Linearly order the simple roots starting from the shortest root $i = 1$. For the triple of roots ($i = 1, j = 2, k = 3$) we get the following equations:

$$\begin{aligned} (10) \quad d_{321} + d_{123} - \phi_{21}(\alpha_2^\vee) - 2\phi_{23}(\alpha_2^\vee) &= d_{312} + d_{213} \\ d_{212} &= \phi_{21}(\alpha_2^\vee) + \phi_{12}(\alpha_1^\vee) \\ d_{121} &= \phi_{12}(\alpha_1^\vee) + 2\phi_{21}(\alpha_2^\vee) \\ d_{323} &= \phi_{32}(\alpha_3^\vee) + \phi_{23}(\alpha_2^\vee). \end{aligned}$$

Let us substitute $\phi_{21}(\alpha_2^\vee)$ from the second and third equations into the first one. Then we obtain

$$d_{321} + d_{123} - d_{121} + d_{212} - 2\phi_{23}(\alpha_2^\vee) = d_{312} + d_{213}.$$

This equation has a solution iff $d_{321} + d_{123} - d_{121} + d_{212} - d_{312} - d_{213}$ is even. Moreover, the whole system (10) has a solution in this case. Note that even if we did not set $\phi_{ij}(\alpha_j^\vee) = 0$ we would still not be able to solve the equation unless the same condition is satisfied. For the other triple of adjacent roots we have to solve the same equations as in the type A case.

Type C_n ($n \geq 3$). Linearly order the simple roots starting from the *longest* root $i = 1$. Then for the triple of roots ($i = 1, j = 2, k = 3$) we get the following equations:

$$\begin{aligned} (11) \quad d_{123} + d_{321} - \phi_{23}(\alpha_2^\vee) - \phi_{21}(\alpha_2^\vee) &= d_{132} + d_{231} \\ d_{121} &= \phi_{12}(\alpha_1^\vee) + \phi_{21}(\alpha_2^\vee) \\ d_{212} &= \phi_{21}(\alpha_2^\vee) + 2\phi_{12}(\alpha_1^\vee) \\ d_{232} &= \phi_{23}(\alpha_2^\vee) + \phi_{32}(\alpha_3^\vee). \end{aligned}$$

Substituting $\phi_{21}(\alpha_2^\vee)$ from the middle two equations into the first one, we obtain

$$d_{123} + d_{321} - \phi_{23}(\alpha_2^\vee) - 2d_{121} + d_{212} = d_{132} + d_{231}.$$

From this equation we determine $\phi_{23}(\alpha_2^\vee)$ and we uniquely determine $\phi_{32}(\alpha_3^\vee)$, $\phi_{12}(\alpha_1^\vee)$ and $\phi_{21}(\alpha_2^\vee)$ from the rest of the equations in the system (11). We proceed with the rest of the diagram as in type A.

Type D_n ($n \geq 4$) or E_n . We order the diagram as in the picture:



The first four roots give the equations

$$\begin{aligned} d_{123} + d_{321} - \phi_{23}(\alpha_2^\vee) - \phi_{23}(\alpha_3^\vee) - \phi_{21}(\alpha_2^\vee) - \phi_{21}(\alpha_1^\vee) &= d_{132} + d_{231} - \phi_{12}(\alpha_2^\vee) - \phi_{32}(\alpha_2^\vee) \\ d_{124} + d_{421} - \phi_{24}(\alpha_2^\vee) - \phi_{24}(\alpha_4^\vee) - \phi_{21}(\alpha_2^\vee) - \phi_{21}(\alpha_1^\vee) &= d_{142} + d_{241} - \phi_{12}(\alpha_2^\vee) - \phi_{42}(\alpha_2^\vee) \\ d_{324} + d_{423} - \phi_{24}(\alpha_2^\vee) - \phi_{24}(\alpha_4^\vee) - \phi_{23}(\alpha_2^\vee) - \phi_{23}(\alpha_3^\vee) &= d_{342} + d_{243} - \phi_{32}(\alpha_2^\vee) - \phi_{42}(\alpha_2^\vee). \end{aligned}$$

If we add the three equations, we see that the system has solutions only if

$$d_{123} + d_{321} - d_{132} - d_{231} + d_{124} + d_{421} - d_{142} - d_{241} + d_{324} + d_{423} - d_{342} - d_{243}$$

is even.

Let us now show that if this expression is even, we indeed have a solution. Set

$$\phi_{12}(\alpha_2^\vee) = \phi_{21}(\alpha_1^\vee) = \phi_{23}(\alpha_3^\vee) = \phi_{32}(\alpha_2^\vee) = \phi_{24}(\alpha_4^\vee) = \phi_{42}(\alpha_2^\vee) = 0.$$

Then the equations become

$$\begin{aligned} (12) \quad d_{123} + d_{321} - \phi_{23}(\alpha_2^\vee) - \phi_{21}(\alpha_2^\vee) &= d_{132} + d_{231} \\ d_{124} + d_{421} - \phi_{24}(\alpha_2^\vee) - \phi_{21}(\alpha_2^\vee) &= d_{142} + d_{241} \\ d_{324} + d_{423} - \phi_{24}(\alpha_2^\vee) - \phi_{23}(\alpha_2^\vee) &= d_{342} + d_{243}. \end{aligned}$$

From the last equation we have

$$\phi_{24}(\alpha_2^\vee) = d_{324} + d_{423} - d_{342} - d_{243} - \phi_{23}(\alpha_2^\vee).$$

Plug it into the second equation:

$$d_{124} + d_{421} - d_{324} - d_{423} + d_{342} + d_{243} + \phi_{23}(\alpha_2^\vee) - \phi_{21}(\alpha_2^\vee) = d_{142} + d_{241}.$$

This and the first equation in (12) have a unique solution under the assumption that the expression written before is even. The tails of the diagram are treated in the same way as in type A.

Type F_4 . We order the four simple roots in the following way:

$$\textcircled{1} \text{---} \textcircled{2} \Rightarrow \textcircled{3} \text{---} \textcircled{4}$$

As in type B, we can solve the system (10) iff $d_{321} + d_{123} - d_{121} + d_{212} - d_{312} - d_{213}$ is even. We get a system of equations identical to (11) for the roots 234, which always has a solution.

Diagram has rank 2. We determine $\phi_{ij}(\alpha_i^\vee)$ and $\phi_{ji}(\alpha_j^\vee)$ from the equations (3), (4), (5).

In this way we have determined $\phi_{ij}(\alpha_k^\vee)$ for any three simple roots unless there is a sub-diagram of type B_3 (in types B and F), D_4 (in types D and E) or G_2 in which case an extra solvability condition is required. \square

Remark. The fact that $\text{CH}^3(G)$ is 2-torsion follows from the following argument ([EKL^V, Proof of Proposition 3.20]). Levine [Lev, Theorem 2.1] shows that under our assumptions $K_0(G) \cong \mathbf{Z}$ with the trivial coniveau (or topological) filtration. There is a surjective map

$$\pi_3: \text{CH}^3(G) \rightarrow \text{gr}^3 K_0(G) \cong 0$$

(see [SGA6, Exp. 0, Théorèmes 2.6, 2.12]) given by sending a cycle to the skyscraper sheaf. There is the third Chern class map c_3 in the other direction

$$c_3: \text{gr}_3 K_0(G) \rightarrow \text{CH}^3(G).$$

By the Grothendieck-Riemann-Roch theorem without denominators we have $c_3 \pi_3 = 2\text{id}$. In particular, the kernel of π_3 , which is the whole $\text{CH}^3(G)$ in our case, is annihilated by 2.

We now proceed to the computation of the cohomology $E_1^{-3,5}$.

Given a cubic form $C(-)$ on a lattice Y , we have the associated quadratic-linear form

$$B(y_1, y_2) = \frac{C(y_2 + y_1) + C(y_2 - y_1)}{2} - C(y_2),$$

which is quadratic in the first argument and linear in the second, and the associated trilinear form

$$T(y_1, y_2, y_3) = C(y_1 + y_2 + y_3) - C(y_1 + y_2) - C(y_2 + y_3) - C(y_1 + y_3) + C(y_1) + C(y_2) + C(y_3).$$

Note that B is an integer-valued form despite the factor of $1/2$.

We have the following relations between B and T :

$$(13) \quad T(y_1, y_1, y_2) = 2B(y_1, y_2), \quad B(y_1 + y_2, y_3) = B(y_1, y_3) + B(y_2, y_3) + T(y_1, y_2, y_3).$$

If $\{\alpha_i^\vee\}_i$ is a basis of Y , the cubic form C is uniquely determined by the values $C(\alpha_i^\vee)$, $B(\alpha_i^\vee, \alpha_j^\vee)$ for $i \neq j$ and $T(\alpha_i^\vee, \alpha_j^\vee, \alpha_k^\vee)$ for i, j, k distinct.

Proposition 2.4. *A cubic form C on the cocharacter lattice Y is W -invariant iff*

- (1) $C(\alpha_i^\vee) = 0$ for all simple coroots α_i^\vee
- (2) $B(\alpha_j^\vee, \alpha_i^\vee) = \alpha_i(\alpha_j^\vee)B(\alpha_i^\vee, \alpha_j^\vee)$ for distinct simple coroots α_i^\vee and α_j^\vee and
- (3)

$$\begin{aligned} 0 &= T(\alpha_i^\vee, \alpha_j^\vee, \alpha_k^\vee), \\ 0 &= \alpha_i(\alpha_j^\vee)B(\alpha_i^\vee, \alpha_k^\vee) + \alpha_i(\alpha_k^\vee)B(\alpha_i^\vee, \alpha_j^\vee) \end{aligned}$$

for distinct simple coroots $\alpha_i^\vee, \alpha_j^\vee, \alpha_k^\vee$.

Proof. A cubic form C is W -invariant iff $C(s_i(y)) = C(y)$ for every $y \in Y$ and every simple reflection s_i . For instance, for $y = \alpha_i^\vee$ we get $C(-\alpha_i^\vee) = C(\alpha_i^\vee)$ which implies that $C(\alpha_i^\vee) = 0$.

Expanding the W -invariance condition, we get

$$\begin{aligned} C(y) &= C(s_i(y)) \\ &= C(y - \alpha_i(y)\alpha_i^\vee) \\ &= C(y) - C(\alpha_i(y)\alpha_i^\vee) + B(y, -\alpha_i(y)\alpha_i^\vee) + B(-\alpha_i(y)\alpha_i^\vee, y). \end{aligned}$$

Since α_i is a non-zero functional, W -invariance is equivalent to $C(\alpha_i^\vee) = 0$ and

$$0 = -B(y, \alpha_i^\vee) + \alpha_i(y)B(\alpha_i^\vee, y).$$

This expression is quadratic in y , so it is enough to check it on $y = a\alpha_j^\vee + b\alpha_k^\vee$ for every integers a and b . Then we get

$$\begin{aligned} 0 &= -B(a\alpha_j^\vee + b\alpha_k^\vee, \alpha_i^\vee) + \alpha_i(y)B(\alpha_i^\vee, a\alpha_j^\vee + b\alpha_k^\vee) \\ &= -B(a\alpha_j^\vee, \alpha_i^\vee) - B(b\alpha_k^\vee, \alpha_i^\vee) - abT(\alpha_j^\vee, \alpha_k^\vee, \alpha_i^\vee) \\ &\quad + (a\alpha_i(\alpha_j^\vee) + b\alpha_i(\alpha_k^\vee))(aB(\alpha_i^\vee, \alpha_j^\vee) + bB(\alpha_i^\vee, \alpha_k^\vee)). \end{aligned}$$

Here we have used the relation (13) between B and T .

This equation has to hold identically in a and b . Therefore, W -invariance is equivalent to the following three equations:

$$\begin{aligned} 0 &= C(\alpha_i^\vee) \\ 0 &= -B(\alpha_j^\vee, \alpha_i^\vee) + \alpha_i(\alpha_j^\vee)B(\alpha_i^\vee, \alpha_j^\vee) \\ T(\alpha_i^\vee, \alpha_j^\vee, \alpha_k^\vee) &= \alpha_i(\alpha_j^\vee)B(\alpha_i^\vee, \alpha_k^\vee) + \alpha_i(\alpha_k^\vee)B(\alpha_i^\vee, \alpha_j^\vee). \end{aligned}$$

From the second equation we see that $B(\alpha_j^\vee, \alpha_i^\vee) = 0$ for orthogonal coroots $\alpha_i^\vee, \alpha_j^\vee$. Since in any triple of distinct simple coroots two are orthogonal, without loss of generality we will assume that i and k are orthogonal. Then from the last equation we get that $T(\alpha_i^\vee, \alpha_j^\vee, \alpha_k^\vee) = 0$ for any triple of distinct simple coroots. \square

We have the following standard fact about cubic invariants which follows from examining the list of fundamental invariants of Weyl groups.

Lemma 2.2. *The abelian group of W -invariant cubic forms on Y is free of rank equal to the number of type A factors in G .*

Our goal will be to show that $E_1^{-3,5}$ is isomorphic to the group of W -invariant cubic forms on Y . We begin by defining a map from $E_0^{-3,5}$ to cubic forms on Y .

Given a collection $\{\phi_{ij}\}_{i,j} \in E_0^{-3,5}$ of elements $\phi_{ij} \in X$, we define a cubic form by

$$C(\alpha_i^\vee) = 0, \quad B(\alpha_i^\vee, \alpha_j^\vee) = \phi_{ij}(\alpha_i^\vee) - \phi_{ji}(\alpha_i^\vee) - \alpha_j(\alpha_i^\vee)\phi_{ij}(\alpha_j^\vee), \quad T(\alpha_i^\vee, \alpha_j^\vee, \alpha_k^\vee) = 0.$$

Proposition 2.5. *This map descends to an isomorphism from $E_1^{-3,5}$ to the group of W -invariant cubic forms on Y .*

Proof. We split the proof of this proposition into four steps.

Step 1. The map annihilates exact elements of $E_0^{-3,5}$.

The collection $\{\phi_{ij}\}_{i,j} \in E_0^{-3,5}$ is exact if there is a collection $\{D_i\}_i \in E_0^{-3,4}$ of elements $D_i \in \wedge^2 X$, such that

$$\phi_{ij}(\alpha_k^\vee) = D_i(\alpha_j^\vee, \alpha_k^\vee) + D_j(\alpha_i^\vee, \alpha_k^\vee) - \alpha_j(\alpha_i^\vee)D_j(\alpha_j^\vee, \alpha_k^\vee).$$

Then

$$\begin{aligned}\phi_{ij}(\alpha_i^\vee) &= D_i(\alpha_j^\vee, \alpha_i^\vee) - \alpha_j(\alpha_i^\vee)D_j(\alpha_j^\vee, \alpha_i^\vee) \\ \phi_{ji}(\alpha_i^\vee) &= D_i(\alpha_j^\vee, \alpha_i^\vee) \\ \phi_{ij}(\alpha_j^\vee) &= D_j(\alpha_i^\vee, \alpha_j^\vee).\end{aligned}$$

Therefore,

$$\begin{aligned}B(\alpha_i^\vee, \alpha_j^\vee) &= \phi_{ij}(\alpha_i^\vee) - \phi_{ji}(\alpha_i^\vee) - \alpha_j(\alpha_i^\vee)\phi_{ij}(\alpha_j^\vee) \\ &= D_i(\alpha_j^\vee, \alpha_i^\vee) - \alpha_j(\alpha_i^\vee)D_j(\alpha_j^\vee, \alpha_i^\vee) - D_i(\alpha_j^\vee, \alpha_i^\vee) - \alpha_j(\alpha_i^\vee)D_j(\alpha_i^\vee, \alpha_j^\vee) \\ &= 0.\end{aligned}$$

In the last line we have used antisymmetry of D_j .

Step 2. The map sends closed elements of $E_0^{-3,5}$ to W -invariant cubic forms. The collection $\{\phi_{ij}\}_{i,j} \in E_0^{-3,5}$ is closed if

$$\begin{aligned}0 &= \phi_{ij}(\alpha_k^\vee) + \phi_{ik}(\alpha_j^\vee) + \phi_{jk}(\alpha_i^\vee) - \alpha_k(\alpha_j^\vee)\phi_{ik}(\alpha_k^\vee) - \alpha_j(\alpha_i^\vee)\phi_{jk}(\alpha_j^\vee) \\ &\quad + (\alpha_k(\alpha_j^\vee)\alpha_j(\alpha_i^\vee) - \alpha_k(\alpha_i^\vee))\phi_{jk}(\alpha_k^\vee).\end{aligned}$$

We have to show that such ϕ_{ij} are sent to cubic forms satisfying

$$\begin{aligned}B(\alpha_j^\vee, \alpha_i^\vee) &= \alpha_i(\alpha_j^\vee)B(\alpha_i^\vee, \alpha_j^\vee) \\ 0 &= \alpha_i(\alpha_j^\vee)B(\alpha_i^\vee, \alpha_k^\vee) + \alpha_i(\alpha_k^\vee)B(\alpha_i^\vee, \alpha_j^\vee).\end{aligned}$$

In other words, we have to show that ϕ_{ij} satisfy

$$\begin{aligned}(14) \quad &\phi_{ji}(\alpha_j^\vee) - \phi_{ij}(\alpha_j^\vee) - \alpha_i(\alpha_j^\vee)\phi_{ji}(\alpha_i^\vee) = \alpha_i(\alpha_j^\vee)(\phi_{ij}(\alpha_i^\vee) - \phi_{ji}(\alpha_i^\vee) - \alpha_j(\alpha_i^\vee)\phi_{ij}(\alpha_j^\vee)) \\ &\alpha_i(\alpha_k^\vee)(\phi_{ij}(\alpha_i^\vee) - \phi_{ji}(\alpha_i^\vee) - \alpha_j(\alpha_i^\vee)\phi_{ij}(\alpha_j^\vee)) = -\alpha_i(\alpha_j^\vee)(\phi_{ik}(\alpha_i^\vee) - \phi_{ki}(\alpha_i^\vee) - \alpha_k(\alpha_i^\vee)\phi_{ik}(\alpha_k^\vee)).\end{aligned}$$

The last equation is automatically satisfied if α_i^\vee is orthogonal to either α_j^\vee or α_k^\vee . Since among three distinct roots two are necessarily orthogonal, we just need to consider the case when α_j^\vee is orthogonal to α_k^\vee . Consider permutations of the indices (i, j, k) in the closedness equation:

$$\begin{aligned}(15) \quad &0 = \phi_{ij}(\alpha_k^\vee) + \phi_{ik}(\alpha_j^\vee) + \phi_{jk}(\alpha_i^\vee) - \alpha_j(\alpha_i^\vee)\phi_{jk}(\alpha_j^\vee) - \alpha_k(\alpha_i^\vee)\phi_{jk}(\alpha_k^\vee) \\ &0 = \phi_{ki}(\alpha_j^\vee) + \phi_{jk}(\alpha_i^\vee) + \phi_{ij}(\alpha_k^\vee) - \alpha_j(\alpha_i^\vee)\phi_{jk}(\alpha_j^\vee) - \alpha_i(\alpha_k^\vee)\phi_{ij}(\alpha_i^\vee) + \alpha_j(\alpha_i^\vee)\alpha_i(\alpha_k^\vee)\phi_{ij}(\alpha_j^\vee) \\ &0 = \phi_{ji}(\alpha_k^\vee) + \phi_{jk}(\alpha_i^\vee) + \phi_{ik}(\alpha_j^\vee) - \alpha_k(\alpha_i^\vee)\phi_{jk}(\alpha_k^\vee) - \alpha_i(\alpha_j^\vee)\phi_{ik}(\alpha_i^\vee) + \alpha_k(\alpha_i^\vee)\alpha_i(\alpha_j^\vee)\phi_{ik}(\alpha_k^\vee) \\ &0 = \phi_{jk}(\alpha_i^\vee) + \phi_{ji}(\alpha_k^\vee) + \phi_{ki}(\alpha_j^\vee) - \alpha_i(\alpha_k^\vee)\phi_{ji}(\alpha_i^\vee) - \alpha_i(\alpha_j^\vee)\phi_{ki}(\alpha_i^\vee).\end{aligned}$$

Taking the sum of the second and third equations and subtracting the first and fourth equations we arrive precisely at the second equation in (14).

The closedness conditions for $k = i$ are

$$\begin{aligned} 0 &= \phi_{ij}(\alpha_i^\vee) - \alpha_j(\alpha_i^\vee)\phi_{ji}(\alpha_j^\vee) + (\alpha_i(\alpha_j^\vee)\alpha_j(\alpha_i^\vee) - 1)\phi_{ji}(\alpha_i^\vee) \\ 0 &= \phi_{ji}(\alpha_j^\vee) - \alpha_i(\alpha_j^\vee)\phi_{ij}(\alpha_i^\vee) + (\alpha_i(\alpha_j^\vee)\alpha_j(\alpha_i^\vee) - 1)\phi_{ij}(\alpha_j^\vee). \end{aligned}$$

The first equation coincides with the first equation in (14).

Step 3. The map from $E_1^{-3,5}$ to W -invariant cubic forms is injective.

Suppose that $B(\alpha_i^\vee, \alpha_j^\vee) = 0$, i.e.

$$\phi_{ij}(\alpha_i^\vee) = \phi_{ji}(\alpha_i^\vee) + \alpha_j(\alpha_i^\vee)\phi_{ij}(\alpha_j^\vee)$$

for every pair of simple coroots $\alpha_i^\vee, \alpha_j^\vee$. Then we need to prove that the corresponding elements ϕ_{ij} are exact, i.e. there are $D_i \in \wedge^2 X$, such that

$$\phi_{ij}(\alpha_k^\vee) = D_i(\alpha_j^\vee, \alpha_k^\vee) + D_j(\alpha_i^\vee, \alpha_k^\vee) - \alpha_j(\alpha_i^\vee)D_j(\alpha_j^\vee, \alpha_k^\vee).$$

First, consider the case of two distinct coroots i, j . Then we get the equations

$$\begin{aligned} \phi_{ij}(\alpha_j^\vee) &= D_j(\alpha_i^\vee, \alpha_j^\vee) \\ \phi_{ij}(\alpha_i^\vee) &= D_i(\alpha_j^\vee, \alpha_i^\vee) + \alpha_j(\alpha_i^\vee)D_j(\alpha_i^\vee, \alpha_j^\vee) \\ \phi_{ji}(\alpha_i^\vee) &= D_i(\alpha_j^\vee, \alpha_i^\vee) \\ \phi_{ji}(\alpha_j^\vee) &= D_j(\alpha_i^\vee, \alpha_j^\vee) + \alpha_i(\alpha_j^\vee)D_i(\alpha_j^\vee, \alpha_i^\vee). \end{aligned}$$

From the first and third equations we determine $D_j(\alpha_i^\vee, \alpha_j^\vee)$ and $D_i(\alpha_j^\vee, \alpha_i^\vee)$. The second and fourth equations will be satisfied if

$$\phi_{ij}(\alpha_i^\vee) = \phi_{ji}(\alpha_i^\vee) + \alpha_j(\alpha_i^\vee)\phi_{ij}(\alpha_j^\vee)$$

and similarly after interchanging i and j . But this equation is exactly $B(\alpha_i^\vee, \alpha_j^\vee) = 0$.

Now consider the case of three distinct coroots (i, j, k) , where we again assume that α_j^\vee and α_k^\vee are orthogonal. The equations we get are

$$\begin{aligned} \phi_{ij}(\alpha_k^\vee) &= D_i(\alpha_j^\vee, \alpha_k^\vee) + D_j(\alpha_i^\vee, \alpha_k^\vee) + \alpha_j(\alpha_i^\vee)D_j(\alpha_k^\vee, \alpha_j^\vee) \\ \phi_{ji}(\alpha_k^\vee) &= D_i(\alpha_j^\vee, \alpha_k^\vee) + D_j(\alpha_i^\vee, \alpha_k^\vee) + \alpha_i(\alpha_j^\vee)D_i(\alpha_k^\vee, \alpha_i^\vee) \\ \phi_{jk}(\alpha_i^\vee) &= -D_k(\alpha_i^\vee, \alpha_j^\vee) - D_j(\alpha_i^\vee, \alpha_k^\vee) \\ \phi_{ik}(\alpha_j^\vee) &= -D_i(\alpha_j^\vee, \alpha_k^\vee) + D_k(\alpha_i^\vee, \alpha_j^\vee) + \alpha_k(\alpha_i^\vee)D_k(\alpha_j^\vee, \alpha_k^\vee) \\ \phi_{ki}(\alpha_j^\vee) &= -D_i(\alpha_j^\vee, \alpha_k^\vee) + D_k(\alpha_i^\vee, \alpha_j^\vee) + \alpha_i(\alpha_k^\vee)D_i(\alpha_j^\vee, \alpha_i^\vee). \end{aligned}$$

Let us substitute $D_j(\alpha_k^\vee, \alpha_j^\vee)$ and its permutations in terms of $\phi_{kj}(\alpha_j^\vee)$ and its permutations:

$$\begin{aligned} \phi_{ij}(\alpha_k^\vee) &= D_i(\alpha_j^\vee, \alpha_k^\vee) + D_j(\alpha_i^\vee, \alpha_k^\vee) + \alpha_j(\alpha_i^\vee)\phi_{kj}(\alpha_j^\vee) \\ \phi_{ji}(\alpha_k^\vee) &= D_i(\alpha_j^\vee, \alpha_k^\vee) + D_j(\alpha_i^\vee, \alpha_k^\vee) + \alpha_i(\alpha_j^\vee)\phi_{ki}(\alpha_i^\vee) \\ \phi_{jk}(\alpha_i^\vee) &= -D_k(\alpha_i^\vee, \alpha_j^\vee) - D_j(\alpha_i^\vee, \alpha_k^\vee) \\ \phi_{ik}(\alpha_j^\vee) &= -D_i(\alpha_j^\vee, \alpha_k^\vee) + D_k(\alpha_i^\vee, \alpha_j^\vee) + \alpha_k(\alpha_i^\vee)\phi_{jk}(\alpha_k^\vee) \\ \phi_{ki}(\alpha_j^\vee) &= -D_i(\alpha_j^\vee, \alpha_k^\vee) + D_k(\alpha_i^\vee, \alpha_j^\vee) + \alpha_i(\alpha_k^\vee)\phi_{ji}(\alpha_i^\vee). \end{aligned}$$

The solvability equation for the first two equations is

$$\phi_{ij}(\alpha_k^\vee) - \alpha_j(\alpha_i^\vee)\phi_{kj}(\alpha_j^\vee) = \phi_{ji}(\alpha_k^\vee) - \alpha_i(\alpha_j^\vee)\phi_{ki}(\alpha_i^\vee).$$

Let's show that it follows from the closedness equations (15). Indeed, equating the second and fourth equations, we get

$$\begin{aligned} & \phi_{ij}(\alpha_k^\vee) - \alpha_j(\alpha_i^\vee)\phi_{jk}(\alpha_j^\vee) - \alpha_i(\alpha_k^\vee)\phi_{ij}(\alpha_i^\vee) + \alpha_j(\alpha_i^\vee)\alpha_i(\alpha_k^\vee)\phi_{ij}(\alpha_j^\vee) \\ &= \phi_{ji}(\alpha_k^\vee) - \alpha_i(\alpha_k^\vee)\phi_{ji}(\alpha_i^\vee) - \alpha_i(\alpha_j^\vee)\phi_{ki}(\alpha_i^\vee). \end{aligned}$$

The terms

$$-\alpha_i(\alpha_k^\vee)\phi_{ij}(\alpha_i^\vee) + \alpha_j(\alpha_i^\vee)\alpha_i(\alpha_k^\vee)\phi_{ij}(\alpha_j^\vee) + \alpha_i(\alpha_k^\vee)\phi_{ji}(\alpha_i^\vee)$$

add up to $-\alpha_i(\alpha_k^\vee)B(\alpha_i^\vee, \alpha_j^\vee) = 0$. The rest coincides with the solvability equation.

Similarly, the solvability equation for the last two equations is satisfied if we repeat the argument with j replaced by k .

Therefore, we just need to be able to solve a smaller system

$$\begin{aligned} \phi_{ij}(\alpha_k^\vee) &= D_i(\alpha_j^\vee, \alpha_k^\vee) + D_j(\alpha_i^\vee, \alpha_k^\vee) + \alpha_j(\alpha_i^\vee)\phi_{kj}(\alpha_j^\vee) \\ \phi_{jk}(\alpha_i^\vee) &= -D_k(\alpha_i^\vee, \alpha_j^\vee) - D_j(\alpha_i^\vee, \alpha_k^\vee) \\ \phi_{ik}(\alpha_j^\vee) &= -D_i(\alpha_j^\vee, \alpha_k^\vee) + D_k(\alpha_i^\vee, \alpha_j^\vee) + \alpha_k(\alpha_i^\vee)\phi_{jk}(\alpha_k^\vee). \end{aligned}$$

This system of equations on $D_i(\alpha_j^\vee, \alpha_k^\vee)$ and its permutations has a (non-unique) solution if the following solvability equation is satisfied:

$$\phi_{ij}(\alpha_k^\vee) + \phi_{jk}(\alpha_i^\vee) + \phi_{ik}(\alpha_j^\vee) = \alpha_j(\alpha_i^\vee)\phi_{kj}(\alpha_j^\vee) + \alpha_k(\alpha_i^\vee)\phi_{jk}(\alpha_k^\vee).$$

This solvability equation is equivalent to the first equation in (15). Therefore, we can find the antisymmetric forms D_i and hence $\{\phi_{ij}\}_{i,j}$ is exact.

Step 4. The map from $E_1^{-3,5}$ to W -invariant cubic forms is surjective.

Without loss of generality, we will assume that G is of type A (see Lemma 2.2). Linearly order the roots and let

$$\phi_{i,i+1}(\alpha_i^\vee) = -\phi_{i+1,i}(\alpha_{i+1}^\vee) = B(\alpha_i^\vee, \alpha_{i+1}^\vee), \quad \phi_{ij}(i) = -\phi_{ij}(j).$$

This is clearly a lift of the cubic form given by B and we just have to show that thus defined $\{\phi_{ij}\}_{i,j}$ is closed. Indeed, we need to check that

$$\phi_{ij}(\alpha_i^\vee) + \phi_{ji}(\alpha_j^\vee) = 0,$$

which is clearly satisfied by assumption, and the closedness equations (15).

The closedness equations are nontrivial only for a triple of consecutive roots $j, i = j + 1$ and $k = j + 2$:

$$\begin{aligned} 0 &= 0 \\ 0 &= \phi_{ij}(\alpha_i^\vee) + \phi_{ij}(\alpha_j^\vee) \\ 0 &= \phi_{ik}(\alpha_i^\vee) + \phi_{ik}(\alpha_k^\vee) \\ 0 &= \phi_{ji}(\alpha_i^\vee) + \phi_{ki}(\alpha_i^\vee). \end{aligned}$$

The first three equations are trivially satisfied from our assumptions and the last one follows from the fact that cubic invariants have $B(\alpha_j^\vee, \alpha_{j+1}^\vee) = B(\alpha_{j+1}^\vee, \alpha_{j+2}^\vee)$. \square

The E_1 page of the spectral sequence is thus

$$\begin{array}{cccc}
E_1^{-3,6} & 0 & 0 & 0 \\
E_1^{-3,5} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & E_1^{-2,3} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & K_3(k)
\end{array}$$

Therefore, the spectral sequence again degenerates at E_1 and we derive the desired cohomology groups.

3. \mathcal{K} -COHOMOLOGY OF REDUCTIVE GROUPS

In this section G will denote a split connected reductive group over a field k with a simply-connected derived group G_{der} . Thus, we have an exact sequence

$$1 \rightarrow G_{der} \rightarrow G \rightarrow H_0 \rightarrow 1$$

which induces an exact sequence of character lattices

$$0 \rightarrow X_0 \rightarrow X \rightarrow X_{der} \rightarrow 0$$

and cocharacter lattices

$$0 \rightarrow Y_{der} \rightarrow Y \rightarrow Y_0 \rightarrow 0.$$

Our goal will be to compute the \mathcal{K}_3 -cohomology of G .

3.1. \mathcal{K}_2 -cohomology. As a warm-up, let's compute the \mathcal{K}_2 -cohomology of G . Recall the E_0 page of the Brylinski–Deligne spectral sequence computing $H^\bullet(G, \mathcal{K}_2)$:

$$\begin{array}{ccc}
\oplus_{W(2)} \mathbf{Z} & 0 & 0 \\
\uparrow & & \\
\oplus_{W(1)} X & 0 & 0 \\
\uparrow & & \\
\wedge^2 X & \oplus_{W(1)} k^\times & 0 \\
& \uparrow & \\
0 & X \otimes k^\times & 0 \\
\\
0 & 0 & K_2(k)
\end{array}$$

By simply-connectivity of G_{der} we can identify $\oplus_{W(1)} \mathbf{Z} \cong X_{der}$. Therefore, the cohomology in the (-1) -st column is

$$E_1^{-1,1} = X_0 \otimes k^\times, \quad E_1^{-1,2} = 0.$$

The differential

$$d: \bigwedge^2 X \rightarrow X_{der} \otimes X$$

is no longer injective, but its kernel is $E_1^{-2,2} = \wedge^2 X_0$.

The differential

$$d: \oplus_{W(1)} X \rightarrow \oplus_{W(2)} \mathbf{Z}$$

is still surjective, so $E_1^{-2,4} = 0$.

We compute the cohomology $E_1^{-2,3}$ in the following way. First, it coincides with the cohomology of

$$\bigwedge^2 X / \bigwedge^2 X_0 \rightarrow X_{der} \otimes X \rightarrow \oplus_{W(2)} \mathbf{Z}.$$

In other words, we have to compute cohomology of the middle column in

$$\begin{array}{ccccccc} 0 & \longrightarrow & X_{der} \otimes X_0 & \longrightarrow & \bigwedge^2 X / \bigwedge^2 X_0 & \longrightarrow & \bigwedge^2 X_{der} \longrightarrow 0 \\ & & \downarrow 1 & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X_{der} \otimes X_0 & \longrightarrow & X_{der} \otimes X & \longrightarrow & X_{der} \otimes X_{der} \longrightarrow 0 \\ & & \downarrow & & \downarrow & \swarrow & \\ & & 0 & & \oplus_{W(2)} \mathbf{Z} & & \end{array}$$

Since the vertical map in the left-most column is an isomorphism, the cohomology in the middle column coincides with the cohomology in the right-most column, which we have already computed. Thus, $E_1^{-2,3}$ coincides with the group of W -invariant quadratic forms on Y_{der} .

The E_1 page of the spectral sequence is

$$\begin{array}{ccccc} E_1^{-2,3} & 0 & 0 \\ \bigwedge^2 X_0 & 0 & 0 \\ 0 & X_0 \otimes k^\times & 0 \\ 0 & 0 & K_2(k) \end{array}$$

The spectral sequence degenerates at E_1 . Denote by $E_n(H_0)$ the same spectral sequence for the group H_0 . As we already know, the spectral sequence is concentrated on the diagonal $p + q = 0$ and, moreover, the pullback map $H^0(H_0, \mathcal{K}_2) \rightarrow H^0(G, \mathcal{K}_2)$ induces isomorphisms on the diagonal terms starting at the E_1 page.

Theorem 3.1 (Brylinski–Deligne). *Let G be a split connected reductive group with a simply-connected derived group. Then the \mathcal{K}_2 -cohomology groups are as follows:*

- $H^0(G, \mathcal{K}_2) \cong H^0(H_0, \mathcal{K}_2)$ is isomorphic to the group of alternating forms A on the cocharacter lattice Y_0 together with a quadratic refinement q and an element of $K_2(k)$,
- $H^1(G, \mathcal{K}_2)$ is isomorphic to the group of W -invariant quadratic forms on Y_{der} .

The higher cohomology groups vanish.

3.2. \mathcal{K}_3 -cohomology. Recall that the E_0 page of the spectral sequence computing $H^\bullet(G, \mathcal{K}_3)$ is

$$\begin{array}{ccccccc}
 & \oplus_{W^{(3)}} \mathbf{Z} & & 0 & & 0 & & 0 \\
 & \uparrow & & & & & & \\
 & \oplus_{W^{(2)}} X & & 0 & & 0 & & 0 \\
 & \uparrow & & & & & & \\
 \oplus_{W^{(1)}} \wedge^2 X & & \oplus_{W^{(2)}} k^\times & & 0 & & 0 & \\
 \uparrow & & \uparrow & & & & & \\
 \wedge^3 X & & \oplus_{W^{(1)}} X \otimes k^\times & & 0 & & 0 & \\
 & & \uparrow & & & & & \\
 0 & & \wedge^2 X \otimes k^\times & & \oplus_{W^{(1)}} K_2(k) & & 0 & \\
 & & & & \uparrow & & & \\
 0 & & 0 & & X \otimes K_2(k) & & 0 & \\
 & & & & & & & \\
 0 & & 0 & & 0 & & K_3(k) &
 \end{array}$$

As for the case of \mathcal{K}_2 -cohomology, we get $E_1^{-1,1} = X_0 \otimes K_2(k)$ and $E_1^{-1,2} = 0$. The cohomology in the (-2) -nd column is

$$E_1^{-2,2} = \wedge^2 X_0 \otimes k^\times, \quad E_1^{-2,4} = 0$$

and $E_1^{-2,3}$ is the group of W -invariant quadratic forms on Y_{der} with values in k^\times .

The cohomology group $E_1^{-3,5}$ is the cohomology of the middle column in

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \oplus_{W^{(1)}} X_{der} \otimes X_0 & \longrightarrow & \oplus_{W^{(1)}} \wedge^2 X / \wedge^2 X_0 & \longrightarrow & \oplus_{W^{(1)}} \wedge^2 X_{der} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \oplus_{W^{(2)}} X_0 & \longrightarrow & \oplus_{W^{(2)}} X & \longrightarrow & \oplus_{W^{(2)}} X_{der} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & \swarrow & \\
 & & 0 & & \oplus_{W^{(3)}} \mathbf{Z} & &
 \end{array}$$

The map

$$\bigoplus_{W^{(1)}} X_{der} \otimes X_0 \longrightarrow \bigoplus_{W^{(2)}} X_0$$

is surjective (see Proposition 2.1). Therefore, the cohomology of the middle column can be computed as the cohomology of the right-most column, which we have shown in the previous section to be isomorphic to the group of W -invariant cubic forms on Y_{der} .

The cohomology group $E_1^{-3,4}$ is the cohomology of the middle column in

$$\begin{array}{ccccccc}
0 & \longrightarrow & K & \longrightarrow & \wedge^3 X / \wedge^3 X_0 & \longrightarrow & \wedge^3 X_{der} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \oplus_{W(1)} X_{der} \otimes X_0 & \longrightarrow & \oplus_{W(1)} \wedge^2 X / \wedge^2 X_0 & \longrightarrow & \oplus_{W(1)} \wedge^2 X_{der} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \oplus_{W(2)} X_0 & \longrightarrow & \oplus_{W(2)} X & \longrightarrow & \oplus_{W(2)} X_{der} \longrightarrow 0
\end{array}$$

The kernel K itself can be written as an extension

$$0 \rightarrow X_{der} \otimes \wedge^2 X_0 \rightarrow K \rightarrow \wedge^2 X_{der} \otimes X_0 \rightarrow 0.$$

Note that the map $K \rightarrow \oplus_{W(1)} X_{der} \otimes X_0$ factors through $K \rightarrow \wedge^2 X_{der} \otimes X_0$.

The cohomology of the right-most column is zero (Proposition 2.2). Therefore, $E_1^{-3,4}$ is the cohomology of

$$\wedge^2 X_{der} \otimes X_0 \rightarrow \oplus_{W(1)} X_{der} \otimes X_0 \rightarrow \oplus_{W(2)} X_0.$$

This complex is X_0 tensored with the (-2) -nd column of the E_0 page of the spectral sequence computing $H^\bullet(G, \mathcal{K}_2)$.

Therefore, $E_1^{-3,4}$ is isomorphic to the group of forms $Q(-, -): Y_{der} \otimes Y_0$ quadratic and W -invariant in the first argument and linear in the second one.

The E_1 page of the spectral sequence is

$E_1^{-3,6}$	0	0	0
$E_1^{-3,5}$	0	0	0
$E_1^{-3,4}$	0	0	0
$\wedge^3 X_0$	$E_1^{-2,3}$	0	0
0	$\wedge^2 X_0 \otimes k^\times$	0	0
0	0	$X_0 \otimes K_2(k)$	0
0	0	0	$K_3(k)$

The only possible differential occurs at the E_1 page $\wedge^3 X_0 \rightarrow E_1^{-2,3}$. To see that it vanishes, observe that the map $H^0(H_0, \mathcal{K}_3) \rightarrow H^0(G, \mathcal{K}_3)$ induces an isomorphism on the diagonal terms of the E_1 pages and the differential vanishes for the H_0 spectral sequence.

We have a possible nontrivial extension

$$1 \rightarrow E_1^{-2,3} \rightarrow H^1(G, \mathcal{K}_3) \rightarrow E_1^{-3,4} \rightarrow 1$$

coming from the filtration on $H^0(H, \mathcal{K}_2)$.

Recall (Proposition 1.2) that $\tilde{H}^0(H, \mathcal{K}_2)$ is the group of alternating forms A on Y together with a quadratic refinement satisfying

$$q(x+y) = q(x)q(y)(-1)^{A(x,y)}.$$

The group $E_1^{-2,3}$ has the forms q restricted to Y_{der} . In contrast, the alternating forms A appearing in $E_1^{-3,4}$ vanish on Y_{der} . This shows the splitting of the exact sequence and hence we get the following result.

Theorem 3.2. *The \mathcal{K}_3 -cohomology of G is as follows:*

- $H^0(G, \mathcal{K}_3) \cong H^0(H_0, \mathcal{K}_3)$ is isomorphic to the direct sum of $K_3(k)$ and the group of cubic forms on Y_0 together with refinements q_1, q_2 obeying the conditions of Proposition 1.3,
- $H^1(G, \mathcal{K}_3)$ consists of quadratic forms on Y_{der} valued in k^\times and forms $Q(-, -)$ on Y_{der} and Y_0 quadratic and W -invariant in the first argument and linear in the second,
- $H^2(G, \mathcal{K}_3)$ is isomorphic to the group of W -invariant cubic forms on Y_{der} ,
- $H^3(G, \mathcal{K}_3)$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^{n_{BDG}}$.

The higher cohomology groups vanish.

4. CENTRAL EXTENSIONS OF GROUPS

4.1. Central extensions. Let $B_\bullet G$ be the simplicial scheme whose space of n -simplices is $G^{\times n}$.

Consider a sheaf \mathcal{F} of abelian groups, such that $H^2(\text{Spec } k, \mathcal{F}) = 0$. In other words, every \mathcal{F} -gerbe on the point is trivializable. Then $H^2(B_\bullet G, \mathcal{F})$ is identified with the group of central extensions of G by \mathcal{F} .

Indeed, an element of $H^2(B_\bullet G, \mathcal{F})$ is an \mathcal{F} -gerbe on the point together with an \mathcal{F} -torsor \mathcal{T} on G and an isomorphism $\text{mult}: m^*\mathcal{T} \rightarrow p_1^*\mathcal{T} \boxtimes p_2^*\mathcal{T}$ on $G \times G$ satisfying an associativity condition on $G \times G \times G$. In other words, this is the data of a *multiplicative torsor* on G .

Similarly, elements of $H^3(B_\bullet G, \mathcal{F})$ are identified with isomorphism classes of *multiplicative gerbes* on G . Recall that a multiplicative gerbe is an \mathcal{F} -gerbe \mathcal{G} on G together with an isomorphism of gerbes $\text{mult}: m^*\mathcal{G} \rightarrow p_1^*\mathcal{G} \boxtimes p_2^*\mathcal{G}$ on $G \times G$ and a natural transformation between two isomorphisms on $G \times G \times G$ obeying an associativity-like condition on $G^{\times 4}$.

The notion of a multiplicative gerbe is equivalent to that of a gerbal central extension of G by $B\mathcal{F}$, but we will be content with the notion of a multiplicative gerbe.

4.2. Cohomology of the classifying space. Cohomology groups $H^\bullet(B_\bullet G, \mathcal{F})$ can be computed via the spectral sequence of a double complex. Its E_1 page is

$$E_1^{p,q} = H^q(G^{\times p}, \mathcal{F})$$

with the E_1 differential $H^q(G^{\times p}, \mathcal{F}) \rightarrow H^q(G^{\times(p+1)}, \mathcal{F})$ given by the alternating sum of the pullbacks along the face maps $G^{\times(p+1)} \rightarrow G^{\times p}$.

We will also need the following computation. Let S_\bullet^1 be the standard simplicial circle whose set of n -simplices consists of $n+1$ elements. Taking the alternating coface complex of the cosimplicial abelian group of functions $\mathbf{Z}[S_\bullet^1]$ on the circle, we get the complex

$$C^\bullet(S^1, \mathbf{Z}) := (\mathbf{Z} \rightarrow \mathbf{Z}^{\oplus 2} \rightarrow \mathbf{Z}^{\oplus 3} \rightarrow \dots)$$

computing the cohomology $H^\bullet(S^1, \mathbf{Z})$.

There is a subcomplex

$$C^\bullet(\text{pt}, \mathbf{Z}) := (\mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{1} \mathbf{Z} \rightarrow \dots)$$

of $C^\bullet(S^1, \mathbf{Z})$ coming from the cohomology of the point and we denote by

$$\tilde{C}^\bullet(S^1, \mathbf{Z}) = (0 \rightarrow \mathbf{Z} \rightarrow \mathbf{Z}^2 \rightarrow \dots)$$

the reduced complex. It computes the reduced cohomology $\tilde{H}^\bullet(S^1, \mathbf{Z})$, which is \mathbf{Z} concentrated in degree 1.

For any sheaf \mathcal{F} of abelian groups we have a map

$$H^m(G, \mathcal{F})^{\oplus n} \rightarrow H^m(G^{\times n}, \mathcal{F})$$

given by the sum of the pullbacks along projections. If we assume that these maps are isomorphisms for each n , the complex $H^m(B_\bullet G, \mathcal{F})$ becomes isomorphic to $H^m(G, \mathcal{F}) \otimes \tilde{C}^\bullet(S^1, \mathbf{Z})$, which has cohomology $H^m(G, \mathcal{F})$ concentrated in degree 1.

4.3. Central extensions of groups by \mathcal{K}_2 . Let us compute the cohomology $H^\bullet(B_\bullet G, \mathcal{K}_2)$. The E_1 page of the spectral sequence is

$$H^1(\text{pt}, \mathcal{K}_2) \rightarrow H^1(G, \mathcal{K}_2) \rightarrow H^1(G \times G, \mathcal{K}_2) \rightarrow \dots$$

$$H^0(\text{pt}, \mathcal{K}_2) \rightarrow H^0(G, \mathcal{K}_2) \rightarrow H^0(G \times G, \mathcal{K}_2) \rightarrow \dots$$

Since there are no linear W -invariant forms on Y_{der} , the group of $W \times W$ -invariant quadratic forms on $Y_{der} \oplus Y_{der}$ is the sum of the groups of W -invariant quadratic forms on each summand. This implies that the map

$$H^1(G, \mathcal{K}_2)^{\oplus n} \rightarrow H^1(G^{\times n}, \mathcal{K}_2)$$

introduced earlier is an isomorphism. Thus the first row of the spectral sequence forms the complex $\tilde{C}^\bullet(S^1, \mathbf{Z}) \otimes H^1(G, \mathcal{K}_2)$, which has cohomology $H^1(G, \mathcal{K}_2)$ in degree 1.

The zeroth row of the spectral sequence is isomorphic to the complex $H^0(B_\bullet H_0, \mathcal{K}_2)$. The cohomology of $H^0(B_\bullet H_0, \mathcal{K}_n)$ has been computed Esnault et al. (see [EKLTV, Theorem 4.6]). Its cohomology in degree m is

$$H^m H^0(B_\bullet H_0, \mathcal{K}_n) \cong \text{Sym}^m(X_0) \otimes K_{n-m}(k).$$

Therefore, the E_2 page of the spectral sequence is

$$\begin{array}{cccc} 0 & H^1(G, \mathcal{K}_2) & 0 & 0 \\ K_2(k) & X_0 \otimes k^\times & \text{Sym}^2(X_0) & 0 \end{array}$$

The spectral sequence degenerates at E_2 . The following statement can be found in [BrD, Theorem 4.7].

Theorem 4.1 (Brylinski–Deligne). *The cohomology $H^\bullet(B_\bullet G, \mathcal{K}_2)$ is given by*

- $H^0(B_\bullet G, \mathcal{K}_2) \cong K_2(k)$,
- $H^1(B_\bullet G, \mathcal{K}_2)$ *is isomorphic to the group of W -invariant linear forms on the cocharacter lattice,*
- $H^2(B_\bullet G, \mathcal{K}_2)$ *is isomorphic to the group of W -invariant quadratic forms on the cocharacter lattice.*

The cohomology vanishes in other degrees.

4.4. Central extensions of groups by \mathcal{K}_3 . Let us repeat the arguments for the cohomology $H^\bullet(B_\bullet G, \mathcal{K}_3)$. In this case the E_1 page of the spectral sequence is

$$H^3(\text{pt}, \mathcal{K}_3) \longrightarrow H^3(G, \mathcal{K}_3) \longrightarrow H^3(G \times G, \mathcal{K}_3) \longrightarrow \dots$$

$$H^2(\text{pt}, \mathcal{K}_3) \longrightarrow H^2(G, \mathcal{K}_3) \longrightarrow H^2(G \times G, \mathcal{K}_3) \longrightarrow \dots$$

$$H^1(\text{pt}, \mathcal{K}_3) \longrightarrow H^1(G, \mathcal{K}_3) \longrightarrow H^1(G \times G, \mathcal{K}_3) \longrightarrow \dots$$

$$H^0(\text{pt}, \mathcal{K}_3) \longrightarrow H^0(H_0, \mathcal{K}_3) \longrightarrow H^0(H_0 \times T_0, \mathcal{K}_3) \longrightarrow \dots$$

Since there are no linear W -invariant forms on Y_{der} , the H^2 row is additive, i.e.

$$E_1^{\bullet, 2} = \tilde{C}^\bullet(S^1, \mathbf{Z}) \otimes H^2(G, \mathcal{K}_3).$$

The number of BDG factors in the decomposition of G is also obviously additive, so the same complex occurs on the third row.

The first row splits as a direct sum of an additive complex representing W -invariant quadratic forms on Y_{der} with values in k^\times and a complex representing quadratic-linear forms on Y_{der} and Y_0 .

The complex computing quadratic-linear forms is quasi-isomorphic to the tensor product of two additive complexes by the Eilenberg-Zilber map. Therefore, it has cohomology concentrated in degree 2.

Therefore, the E_2 page of the spectral sequence is

$$\begin{array}{cccc} 0 & H^3(G, \mathcal{K}_3) & 0 & 0 \\ 0 & H^2(G, \mathcal{K}_3) & 0 & 0 \\ 0 & E_2^{1,1} & E_2^{2,1} & 0 \\ K_3(k) & X_0 \otimes K_2(k) & \text{Sym}^2(X_0) \otimes k^\times & \text{Sym}^3(X_0) \end{array}$$

There is a possible E_2 differential

$$E_2^{1,1} \rightarrow \text{Sym}^3(X_0).$$

To see that it is zero, consider the pullback map

$$H^\bullet(B_\bullet H_0, \mathcal{K}_3) \rightarrow H^\bullet(B_\bullet G, \mathcal{K}_3).$$

If a \mathcal{K}_3 -multiplicative gerbe is trivial on G , it is necessarily trivial on H_0 since $G \rightarrow H_0$ is surjective. Therefore, the pullback map is injective. In particular, $\text{Sym}^3(X_0) \rightarrow E_\infty^{3,0}$ is injective. But since $E_\infty^{3,0}$ is a quotient of $\text{Sym}^3(X_0)$, the E_2 differential vanishes and the spectral sequence degenerates at E_2 .

Theorem 4.2. *The cohomology groups $H^\bullet(B_\bullet G, \mathcal{K}_3)$ are given as follows:*

- $H^0(B_\bullet G, \mathcal{K}_3) \cong K_3(k)$,
- $H^1(B_\bullet G, \mathcal{K}_3)$ is isomorphic to the group of W -invariant linear forms on the cocharacter lattice valued in $K_2(k)$
- $H^2(B_\bullet G, \mathcal{K}_3)$ is isomorphic to the group of W -invariant quadratic forms on the cocharacter lattice valued in k^\times ,

- $H^3(B_\bullet G, \mathcal{K}_3)$ is isomorphic to the group of W -invariant cubic forms on the cocharacter lattice,
- $H^4(B_\bullet G, \mathcal{K}_3)$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^{n_{\text{BDG}}}$.

The other cohomology groups vanish.

Using the fact that cohomology of the classifying space parametrizes central extensions, we conclude the following facts:

- (1) Central extensions of G by \mathcal{K}_3 are parametrized by W -invariant quadratic forms on the cocharacter lattice valued in k^\times .
- (2) Gerbal central extensions of G by \mathcal{K}_3 are parametrized by W -invariant cubic forms on the cocharacter lattice.

4.5. Extensions of loop groups. Let us explain how thus obtained central extensions of groups by \mathcal{K}_2 and \mathcal{K}_3 relate to central extensions of loop and double loop groups. This is an algebraic counterpart [KV, Section 4.3] of transgression of central extensions to smooth loop groups, see e.g. [Br, Section 6]. See also [Wal] for an explicit relation between multiplicative gerbes on G and central extensions of the loop group.

For a space X over a field k (i.e. a sheaf over the category of affine schemes over k) we denote

$$LX = \text{Map}(\text{Spec } k((t)), X)$$

the loop space of X .

Consider a correspondence

$$\begin{array}{ccc} & LX \times D^\times & \\ \pi \swarrow & & \searrow \text{ev} \\ LX & & X \end{array}$$

Given an element $t \in H^n(X, \mathcal{K}_m)$ its pullback ev^*t defines an element of $H^n(LX \times D^\times, \mathcal{K}_m)$. The residue map on K-theory defines a map

$$\pi_* : H^n(LX \times D^\times, \mathcal{K}_m) \rightarrow H^n(LX, \mathcal{K}_{m-1}).$$

Thus, we have defined a transgression map

$$\pi_* \text{ev}^* : H^n(X, \mathcal{K}_m) \rightarrow H^n(LX, \mathcal{K}_{m-1})$$

which sends an $(n-1)$ -gerbe over \mathcal{K}_m on X to an $(n-1)$ -gerbe over \mathcal{K}_{m-1} on LX .

This behaves well with respect to multiplicative structures. Indeed,

$$LB_\bullet G \cong B_\bullet LG.$$

Therefore, we get a map

$$H^{n+1}(B_\bullet G, \mathcal{K}_m) \rightarrow H^{n+1}(B_\bullet LG, \mathcal{K}_{m-1})$$

which sends multiplicative $(n-1)$ -gerbe over \mathcal{K}_m on G to a multiplicative $(n-1)$ -gerbe over \mathcal{K}_{m-1} on LG .

For instance, a multiplicative \mathcal{K}_2 -torsor on G gives rise to a \mathbf{G}_m -central extension of LG . In this way we get a central extension of LG by \mathbf{G}_m for any W -invariant quadratic form on the cocharacter lattice.

The results of this paper produce a \mathbf{BG}_m -central extension of $L^2G \equiv LLG$, the double loop group, given any W -invariant cubic form on the cocharacter lattice. Similarly, we get a \mathbf{G}_m -central extension of L^2G given any W -invariant quadratic form on the cocharacter lattice valued in k^\times . We expect that the gerbal central extension of L^2GL_n constructed in [AK] and [FZ] come from the basic W -invariant cubic form.

REFERENCES

- [AK] S. Arkhipov, K. Kremnizer, 2-gerbes and 2-Tate spaces, *Arithmetic and Geometry around Quantization*, Progr. in Math. **279** (2010) 23–35, [arXiv:0708.4401](#).
- [Br] J.-L. Brylinski, *Loop Spaces, Characteristic Classes and Geometric Quantization*, Progr. in Math. **107** (1993) Birkhäuser.
- [BrD] J.-L. Brylinski, P. Deligne, Central extensions of reductive groups by \mathcal{K}_2 , *Publ. Math. IHES* **94** (2001) 5–85.
- [EKL] H. Esnault, B. Kahn, M. Levine, E. Viehweg, The Arason invariant and mod 2 algebraic cycles, *J. of AMS* **11** (1998) 73–118.
- [FZ] E. Frenkel, X. Zhu, Gerbal representations of double loop groups, *Int. Math. Res. Notices* **2012** 3929–4013, [arXiv:0810.1487](#).
- [Gil] S. Gille, The first Suslin homology sheaf of a split simply connected semisimple algebraic group, *J. of Algebra* **333** (2011) 26–39.
- [Gr] A. Grothendieck, Torsion homologique et sections rationnelles, *Sém. Claude Chevalley* **3** (1958) exp. 5.
- [Kac] V. Kac, Torsion in cohomology of compact Lie groups and Chow rings of reductive algebraic groups, *Invent. Math.* **80** (1985) 69–79.
- [KN] S. Kaji, M. Nakagawa, The Chow rings of algebraic groups E_6 , E_7 and E_8 , [arXiv:0709.3702](#).
- [KV] M. Kapranov, E. Vasserot, Formal loops II: A local Riemann–Roch theorem for determinantal gerbes, *Ann. Sci. Éc. Norm. Supér.* **40** (2007) 113–133, [arXiv:math/0509646](#).
- [Lev] M. Levine, The algebraic K-theory of the classical groups and some twisted forms, *Duke Math. J.* **70** (1993) 405–443.
- [Mar] R. Marlin, Anneaux de Chow des groupes algébriques $SO(n)$, $Spin(n)$, G_2 et F_4 , *Publ. Math. d’Orsay*, **95** (1974).
- [SGA6] SGA6: *Théorie des intersections et théorème de Riemann-Roch*, dirigé par P. Berthelot, A. Grothendieck, L. Illusie, *Lect. Notes in Math.* **225**, Springer-Verlag (Berlin, 1971).
- [SGA7] SGA7: *Groupes de monodromie en géométrie algébrique* par A. Grothendieck, *Lect. Notes in Math.* **288** (1972) Springer-Verlag.
- [Wal] K. Waldorf, Transgressive loop group extensions, [arXiv:1502.05089](#).

MATHEMATICAL INSTITUTE, RADCLIFFE OBSERVATORY QUARTER, WOODSTOCK ROAD, OXFORD, OX2 6GG, UK

E-mail address: safronov@maths.ox.ac.uk